(1) Evaluate $\displaystyle\iint_R \frac{xy}{\sqrt{x^2+y^2+1}}\,dA \quad R:\{(x,y): 0 \le x \le 1,\ 0 \le y \le 1\}$.

$$\iint_R \frac{xy}{\sqrt{x^2+y^2+1}}\,dA = \int_0^1\int_0^1 \frac{xy}{\sqrt{x^2+y^2+1}}\,dy\,dx \qquad \begin{aligned} u &= x^2+y^2+1 \\ du &= 2y\,dy \end{aligned}$$

$$= \int_0^1 \int_{x^2+1}^{x^2+2} \frac{x}{2}\, u^{-1/2}\,du\,dx = \int_0^1 x u^{1/2}\Big]_{x^2+1}^{x^2+2} dx$$

$$= \int_0^1 \left(x\sqrt{x^2+2} - x\sqrt{x^2+1}\right) dx$$

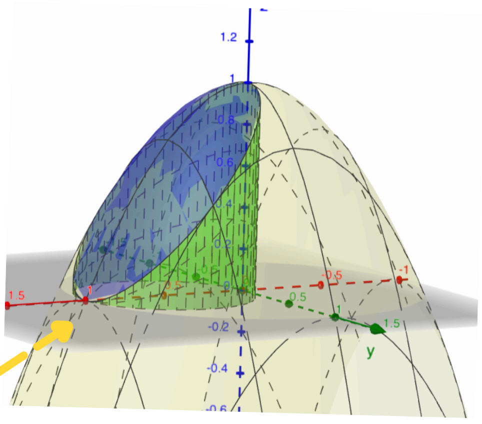
$$u = x^2+2 \qquad\qquad u = x^2+1$$

$$= \frac{1}{2}\int_2^3 u^{1/2}\,du - \frac{1}{2}\int_1^2 u^{1/2}\,du$$

$$= \frac{1}{3}\left(u^{3/2}\Big]_2^3 - \frac{1}{3}u^{3/2}\Big]_1^2\right)$$

$$= \frac{1}{3}\left(3^{3/2} - 2^{3/2} - 2^{3/2} + 1\right)$$

$$\boxed{= \frac{1}{3}\left(3\sqrt{3} - 4\sqrt{2} + 1\right)}$$



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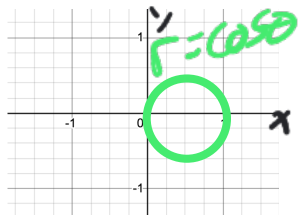
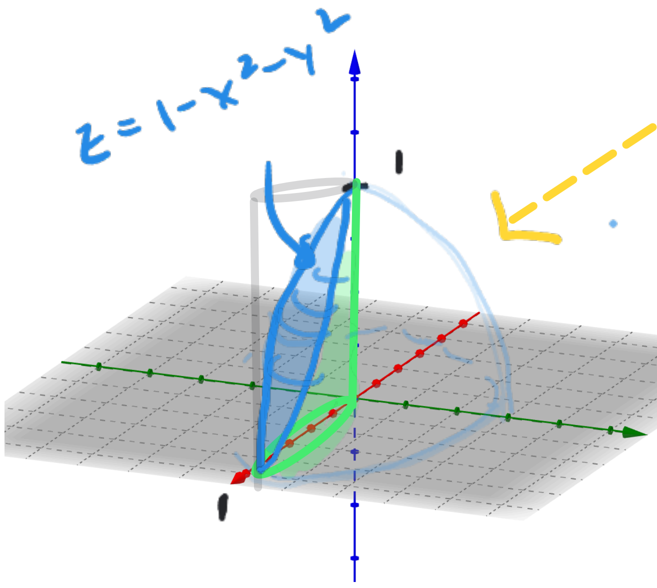
(2) SET UP BUT DO NOT EVALUATE: Use a double integral in polar coordinates to find the volume of the solid bounded above by the surface $z = 1 - x^2 - y^2$, below by the xy plane, and laterally by the cylinder $x^2 + y^2 - x = 0$.



$$x^2 - x + \tfrac{1}{4} + y^2 = 0 + \tfrac{1}{4}$$

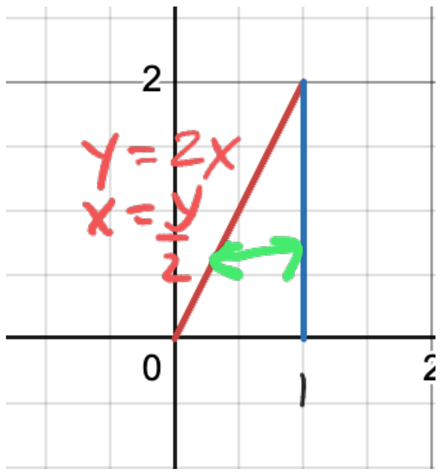
$$\left(x - \tfrac{1}{2}\right)^2 + y^2 = \tfrac{1}{4}$$

This is an off center circle - a common polar curve (you can get by converting) $r = \cos\theta$, which is swept out for $0 \le \theta \le \pi$ (review polar)



$$V = \int_0^{\pi}\int_0^{\cos\theta} (1-r^2)\,r\,dr\,d\theta$$

(3) Evaluate $\int_0^2 \int_{y/2}^1 \cos(x^2)\,dx\,dy$



Cannot integrate cos(x²) (without series) so change order

$$\int_0^2 \int_{y/2}^1 \cos(x^2)\,dx\,dy = \int_0^1 \int_0^{2x} \cos(x^2)\,dy\,dx = \int_0^1 2x\cos(x^2)\,dx = \sin(1)$$

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(4)

(a) Convert $(4, \pi/2, 3)$ from cylindrical coordinates to

$r \quad \theta \quad z$

$x = r\cos\theta = 4\cos\frac{\pi}{2}$

$y = r\sin\theta = 4\sin\frac{\pi}{2}$

rectangular coordinates $(0, 4, 3)$ Ans: $(0,4,3)$

$\rho^2 = 0^2 + 4^2 + 3^2 \quad \rho = 5$

$\rho\cos\phi = z$

$\cos\phi = \frac{z}{\rho} = 3/5$

spherical coordinates $\left(5, \frac{\pi}{2}, \cos^{-1}\frac{3}{5}\right)$ Ans: $(5, \pi/2, \cos^{-1}(3/5))$

(b) Convert $(2, 2, \sqrt{2})$ from rectangular coordinates to

$r^2 = x^2 + y^2 \quad r = \sqrt{8}$

$\tan\theta = \frac{y}{x} = 1$

cylindrical coordinates $(2\sqrt{2}, \pi/4, \sqrt{2})$ Ans: $(2\sqrt{2}, \pi/4, \sqrt{2})$

$\rho^2 = x^2 + y^2 + z^2$
$= 4 + 4 + 2$

$\cos\phi = \frac{z}{\rho} = \frac{\sqrt{2}}{\sqrt{10}}$

spherical coordinates $\left(\sqrt{10}, \frac{\pi}{4}, \cos^{-1}\left(\frac{1}{\sqrt{5}}\right)\right)$ Ans: $(\sqrt{10}, \pi/4, \cos^{-1}(1/\sqrt{5}))$

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(5) SET UP BUT DO NOT EVALUATE: $\iiint_E f(x,y,z)\,dV$ where E is the solid bounded by the paraboloid $z = 4x^2 + y^2$ and the parabolic cylinder $z = 4 - 3y^2$.

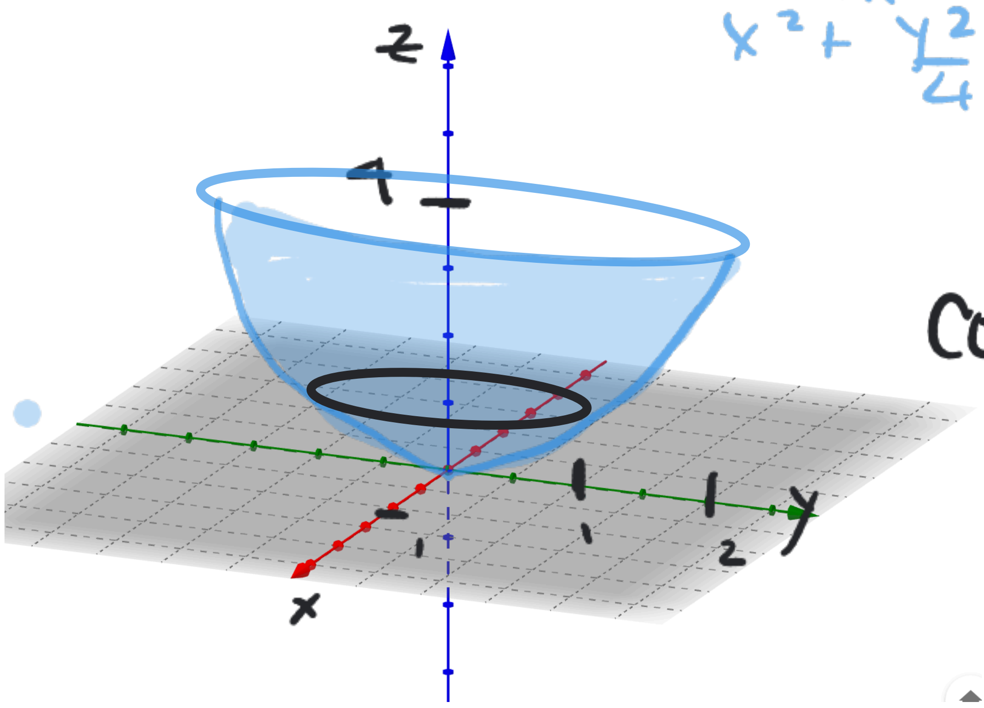
Sketch

Trace $z=4$

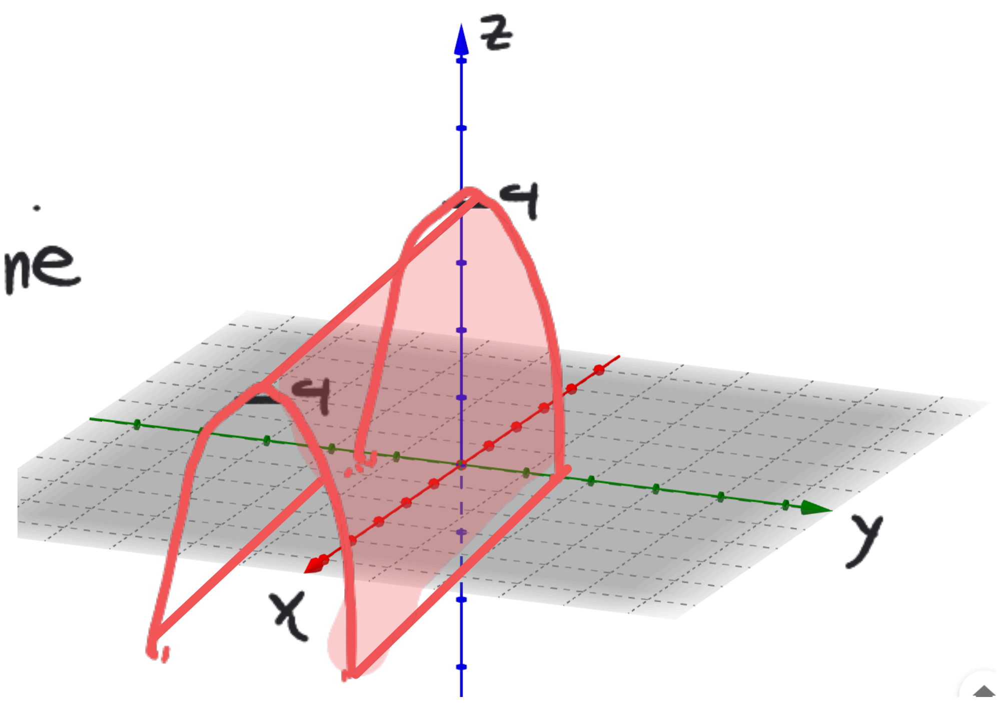
$4=4x^2+y^2$

$x^2+\frac{y^2}{4}=1$

$z=4-3y^2$



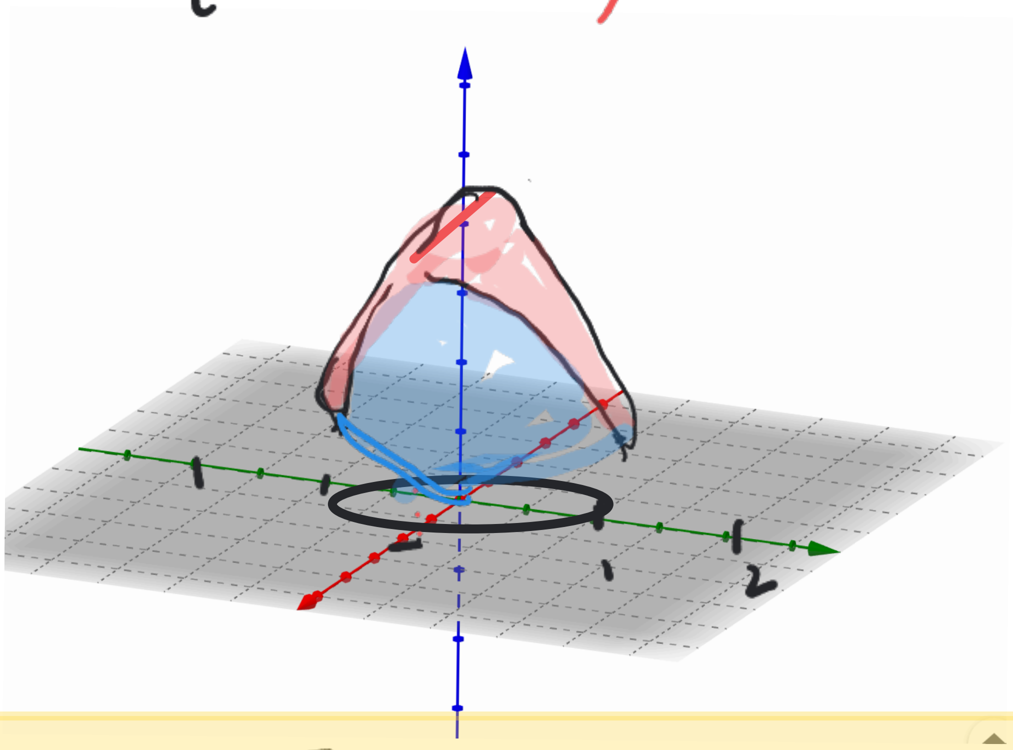
Combine

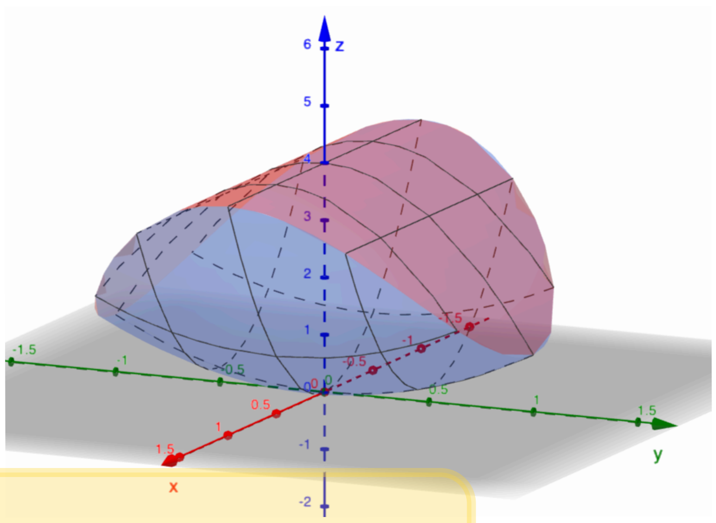


Intersection / projection xy plane

$$\begin{cases} z=4x^2+y^2 \\ z=4-3y^2 \end{cases} \Rightarrow 4x^2+y^2=4-3y^2 \Rightarrow 4x^2+4y^2=4$$

$$x^2+y^2=1$$





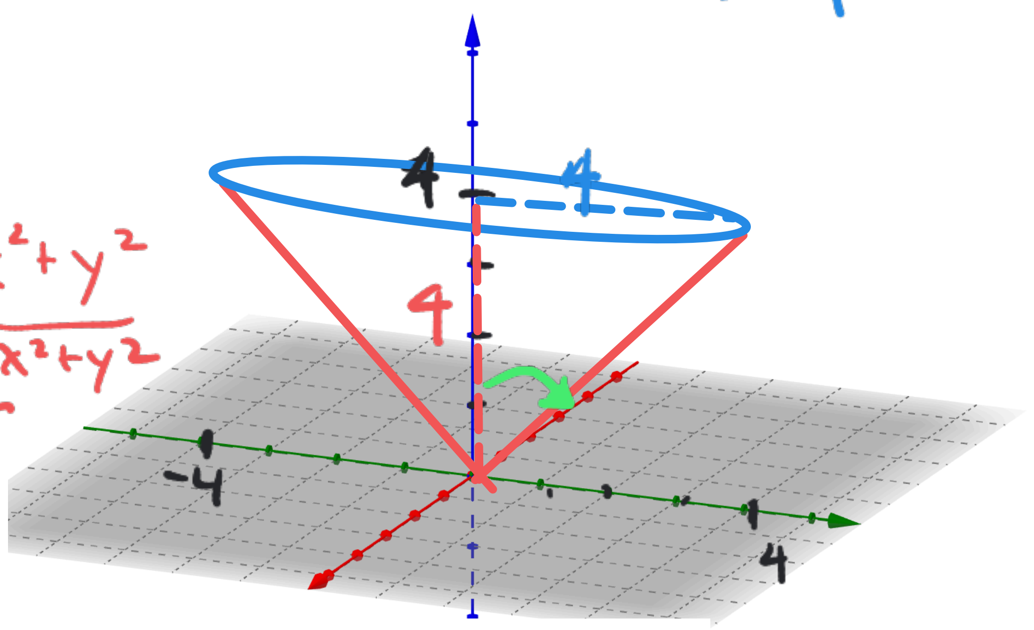
$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{4x^2+y^2}^{4-3y^2} f(x,y,z)\,dz\,dy\,dx$$

(6) SET UP BUT DO NOT EVALUATE: integrals as specified to find the volume enclosed by the cone $z^2 = x^2 + y^2$ and the plane $z = 4$.

Intersection
$x^2+y^2=16$

$z=4$

$z^2=x^2+y^2$
$z=\sqrt{x^2+y^2}$
$z=r$



a) Triple integral - cylindrical coordinates.

Ans: $$\int_0^{2\pi}\int_0^4\int_r^4 r\,dz\,dr\,d\theta$$

b) Triple integral - spherical coordinates.

Convert $z=4$ to spherical
$\rho\cos\phi=4$ $\rho=$

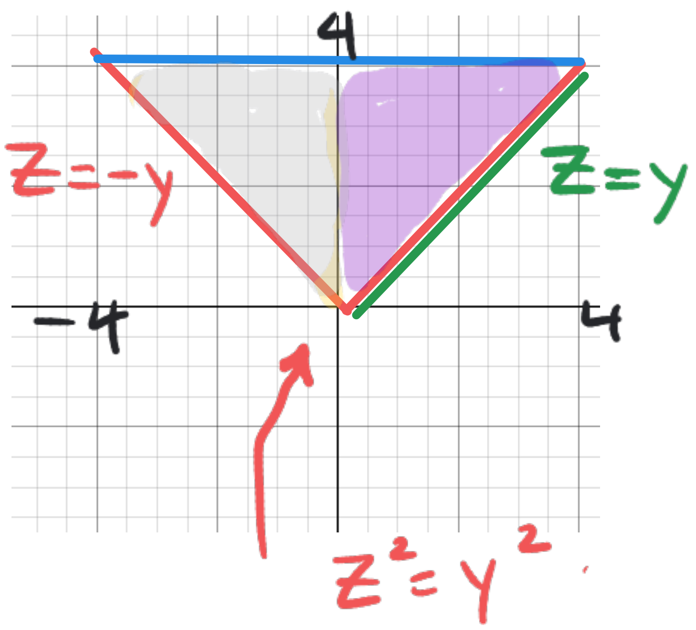
Ans: $$\int_0^{2\pi}\int_0^{\pi/4}\int_0^{4\sec\phi} \rho^2\sin\phi\,d\rho\,d\varphi\,d\theta$$

d) Triple integral- rectangular coordinates; order dx dz dy

c) Double integral - rectangular coordinates ; order dy dx .

$$\int_{-4}^{4}\int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}}\left(4-\sqrt{x^2+y^2}\right)dy\,dx$$

d) Triple integral- rectangular coordinates; order dx dz dy



$$\int_{-4}^{0}\int_{-y}^{4}\int_{-\sqrt{z^2-y^2}}^{\sqrt{z^2-y^2}} dx\,dz\,dy \; + \int_{0}^{4}\int_{y}^{4}\int_{-\sqrt{z^2-y^2}}^{\sqrt{z^2-y^2}} dx\,dz\,dy$$

(7) Given $\displaystyle\int_0^{2\pi}\int_0^{1}\int_0^{\sqrt{4-r^2}} r\,dz\,dr\,d\theta$

$\underbrace{\int_0^{2\pi}\int_0^{1}}_{\text{Unit circle}}$; $\sqrt{4-r^2}$ → Top

$z=\sqrt{4-r^2}=\sqrt{4-x^2-y^2}$

$z^2=4-x^2-y^2$

$x^2+y^2+z^2=4$

⇒ solid is below the piece of the sphere, over the unit circle

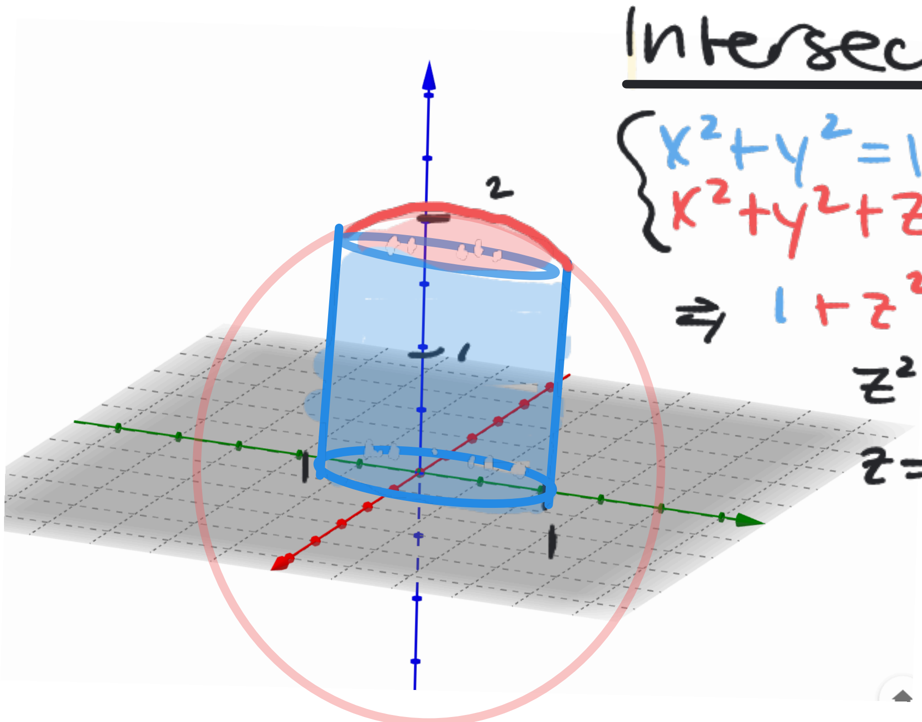
Intersection

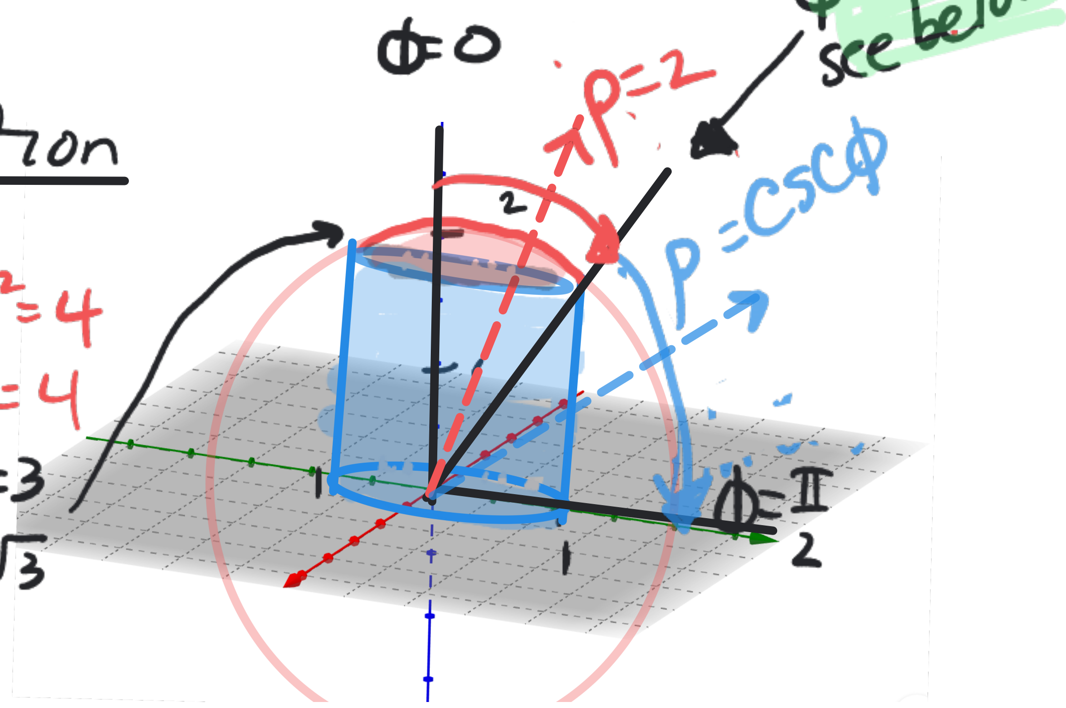
$\begin{cases} x^2+y^2=1 \\ x^2+y^2+z^2=4 \end{cases}$

$\Rightarrow 1+z^2=4$

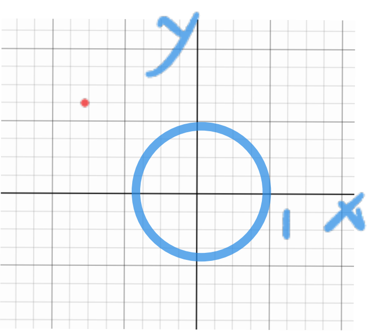
$z^2=3$

$z=\sqrt{3}$





b) Convert the triple integral to <u>Rectangular Coordinates</u>: DO NOT EVALUATE.

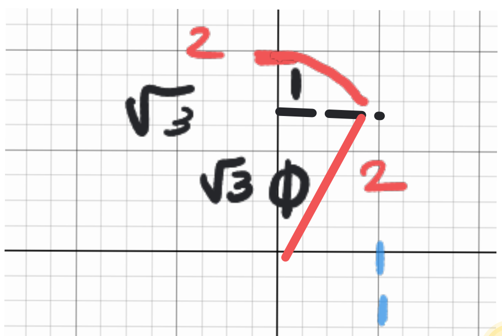


$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_0^{\sqrt{4-x^2-y^2}} dz\,dy\,dx$$

c) Convert the triple integral to <u>Spherical Coordinates</u>: DO NOT EVALUATE. see sketch

Need to split integral in two since for $0\le\phi\le$ ρ goes through sphere $\rho=2$ and for $\phi\le\frac{\pi}{2}$, ρ goes through cylinder $x^2+y^2=1$ which we convert to spherical $r=1 \Rightarrow \rho\sin\phi=1$

$\rho=\frac{1}{\sin\phi}=\csc\phi$



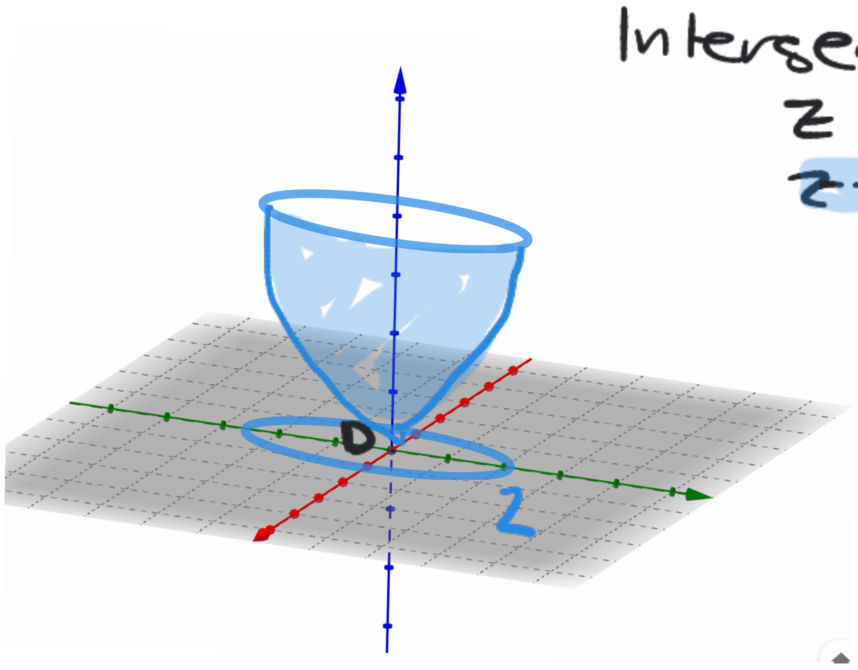
$\tan\phi=1/\sqrt{3}$

$\phi=\pi/6$

$$\int_0^{2\pi}\int_0^{\pi/6}\int_0^{2}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta+\int_0^{2\pi}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\int_0^{\csc\phi}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

(8) Evaluate $\iint_S z\,dS$ where S is the portion of the paraboloid $z=x^2+y^2$ that lies under the plane $z=4$.

Ans: $\frac{\pi}{60}\left(391\sqrt{17}+1\right)$



Intersection

$z=4$
$z=x^2+y^2$ $\Rightarrow$ $x^2+y^2=4$

Surface $z=x^2+y^2=g(x,y)$

so $dS=\sqrt{g_x^2+g_y^2+1}\,dA$

$dS=\sqrt{4x^2+4y^2+1}\,dA$

$z=x^2+y^2$

$$\iint_S z\,dS = \iint_D (x^2+y^2)\sqrt{4x^2+4y^2+1}\,dA$$

$$=\int_0^{2\pi}\int_0^2 r^2\sqrt{4r^2+1}\;r\,dr\,d\theta$$

$u=4r^2+1$
$du=8r\,dr$

$$=\int_0^{2\pi}\int_1^{17}\frac{u-1}{4}\sqrt{u}\;\frac{du}{8}\;d\theta$$

$\frac{102}{3}-\frac{130}{3}=-\frac{28}{3}$

$$=\frac{1}{32}\cdot 2\pi\int_1^{17}\left(u^{3/2}-u^{1/2}\right)du=\frac{\pi}{16}\left(\frac{2}{5}u^{5/2}-\frac{2}{3}u^{3/2}\right]_1^{17}$$

$$=\frac{\pi}{16}\cdot 2u^{3/2}\left(\frac{1}{5}u-\frac{1}{3}\right]_1^{17}=\frac{\pi}{8}\left[17^{3/2}\left(\frac{17}{5}-\frac{1}{3}\right)-\left(\frac{1}{5}-\frac{1}{3}\right)\right]$$

$$=\frac{\pi}{8}\left(17^{3/2}\cdot\frac{46}{15}+\frac{2}{15}\right)=\frac{\pi}{8}\cdot\frac{2}{15}\left(17^{3/2}(23)+1\right)=\boxed{\frac{\pi}{60}\left(391\sqrt{17}+1\right)}$$

(9) Omit

(10) $\int_C yz\cos x\, ds$ where C is given by x=t, y=2cost, z=2sint, $0 \le t \le \pi$. Ans: $\frac{8\sqrt{5}}{3}$

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

$$= \sqrt{1 + (-4\sin t)^2 + (4\cos t)^2}\, dt$$

$$= \sqrt{5}\, dt$$

$$\int_C yz\cos x\, ds = \int_0^{\pi} 2\cos t\, 2\sin t\, \cos t\, \sqrt{5}dt$$

$$= 4\sqrt{5}\int_0^{\pi} \cos^2 t \sin t\, du \qquad u = \cos t$$

$$du = -\sin t dt$$

$$= -4\sqrt{5}\int_1^{-1} u^2\, du$$

$$= -4\sqrt{5}\ \frac{u^3}{3}\Big]_1^{-1} = -4\sqrt{5}\left(-\frac{1}{3} - \frac{1}{3}\right)$$

$$= \boxed{\frac{8\sqrt{5}}{3}}$$